

# Fidelity estimation between two finite ensembles of unknown pure equatorial qubit states

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## Abstract

Suppose, we are given two finite ensembles of pure qubit states, so that the qubits in each ensemble are prepared in identical (but unknown for us) states lying on the equator of the Bloch sphere. What is the best strategy to estimate fidelity between these two finite ensembles of qubit states? We discuss three possible strategies for the fidelity estimation. We show that the best strategy includes two stages: a specific unitary transformation on two ensembles and state estimation of the output states of this transformation.

*Keywords:* State reconstruction, quantum state engineering and measurements, optimal quantum transformations, quantum information

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The state of a quantum system can be perfectly reconstructed only by computing statistical averages of different observables on a large ensemble of identically prepared systems. In practice, however, we are usually given with a very limited number of the identical copies. Any measurement at such limited ensemble provides us with partial information about the state of the system. This leads to an important problem of the optimal extraction of information from finite ensembles of quantum systems [1, 2].

In this paper we focus on the best strategy for extraction of information about fidelity between two finite ensembles of unknown equatorial qubit states. In order to simplify our discussion we assume at the moment that each ensemble contains  $N$  separable particles initially prepared in pure states  $|\psi_a\rangle$  and  $|\psi_b\rangle$ . A pure equatorial qubit state can be parameterized with a

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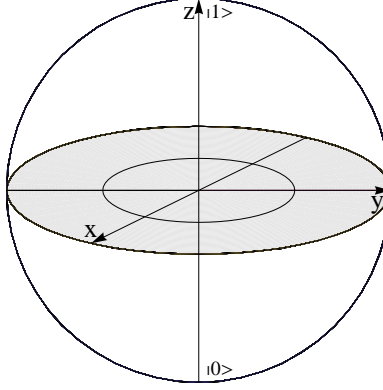


Figure 1: Bloch sphere representation of a qubit state.

single parameter as

$$|\psi_k\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\phi_k} |1\rangle) , \quad (1)$$

where  $k$  stands for subindexes  $\{a, b\}$  which refer to the different ensembles and  $\{|0\rangle, |1\rangle\}$  is a computational basis. A pure equatorial state (1) can be visualized as a point lying on a big circle which is formed by the intersection of the Bloch sphere with  $x - y$  plane. All pure equatorial states are displayed in Fig. 1 with the big circle.

What is the best strategy to estimate the fidelity

$$F_{a,b} \equiv |\langle \psi_a | \psi_b \rangle|^2 = \frac{1}{4} |1 + e^{i(\phi_b - \phi_a)}|^2 \quad (2)$$

between the finite ensembles of equatorial qubit states  $|\psi_a\rangle$  and  $|\psi_b\rangle$ ? This question touches the very foundation of quantum information theory and takes its place among such widely discussed problems as the state estimation [1, 2], the state discrimination [3, 4] and the state comparison [5]. Apart of academic interest, moreover, the fidelity estimation problem may be relevant in implementation of schemes for quantum communication with linearly polarized photons and for linear optics quantum computation [6]. For example, we are given with a finite ensemble of  $2N$  identical linearly polarized photons. Each photon in the ensemble can be described by some quantum state  $|\psi_a\rangle$ . Suppose that a half of the photons from the ensemble is subjected independently to some unitary evolution so that the outputs are in the state  $|\psi_b\rangle$ .

We like to know the effect of the unitary evolution by comparing the phases of the states  $|\psi_a\rangle$  and  $|\psi_b\rangle$ . This effect can be quantified with the fidelity (2).

The simplest strategy to estimate the fidelity between the ensembles of states  $|\psi_a\rangle$  and  $|\psi_b\rangle$  is to perform state estimation of each of these states independently on each other and compute the fidelity (2) between the estimated states  $|\psi'_a\rangle$  and  $|\psi'_b\rangle$ . Such a strategy for the fidelity estimation may be called the measurement-based. An optimal scheme to estimate the state of equatorial qubits being given  $N$  identical replicas was proposed by Derka *et al.* [2]. In this scheme, in more details, a positive operator valued measurement (POVM), which is characterized by a set of orthogonal projectors, need to be performed on the composite system of all  $N$  qubits. Since the state of the  $N$ -qubit system always remains within the totally symmetric subspace of  $\mathcal{H}_2^{\otimes N}$  where  $\mathcal{H}_2$  is the two-dimensional qubit state space, the dimensionality of the space in which the POVM need to be defined is  $N + 1$ . If  $|n\rangle$ ,  $n = 0, \dots, N$  is an orthonormal basis in this  $N + 1$ -dimensional space, the optimal POVM for the state estimation of equatorial qubit is given by the set of  $k = 1, \dots, N$  orthogonal projectors  $P_k = |\Psi_k\rangle \langle \Psi_k|$  where

$$|\Psi_k\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^N e^{i \frac{2\pi}{N+1} k n} |n\rangle. \quad (3)$$

Within this scheme, the maximal mean fidelity between the original state  $|\psi_a\rangle$  and the reconstructed state  $|\psi'_a\rangle$  is given by

$$\bar{f}(|\psi_a\rangle, |\psi'_a\rangle) = \frac{1}{2} + \frac{1}{2^{N+1}} \sum_{i=0}^{N-1} \sqrt{C_i^N C_{i+1}^N}, \quad (4)$$

where  $C_i^N$  and  $C_{i+1}^N$  denote the binomial coefficients. Later we shall always use term “probability” instead of mean fidelity between estimated and actual values in order to avoid any confusion. Term “fidelity” will be used only with regard to the value (2) of interest. As indicated above, the scheme for the state estimation of equatorial qubits can be employed to estimate fidelity between two finite ensembles of equatorial qubit states. Since the states of interest are estimated independently, the probability to reconstruct fidelity (2) correctly is given by  $\bar{f}^2(|\psi_a\rangle, |\psi'_a\rangle)$  and displayed in Fig. 2 by dots.

An alternative strategy for the fidelity estimation is based on quantum cloning and includes two stages. At the first stage we provide infinite many copies from available replicas of the unknown states  $|\psi_a\rangle$  and  $|\psi_b\rangle$ . This task

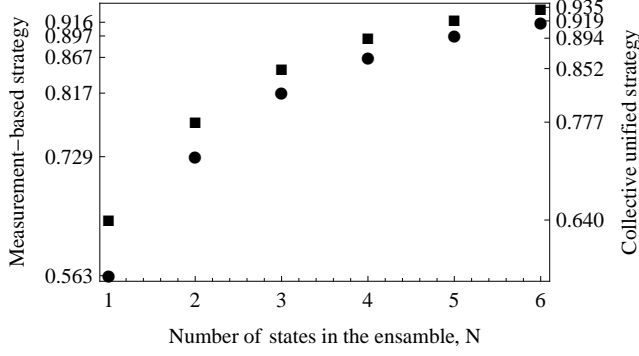


Figure 2: The probabilities to reconstruct fidelity (2) by the first measurement-based strategy (dots) and the third collective unified strategy (squares).

can be realized with  $N \rightarrow \infty$  equatorial quantum cloning machine (EQCM) [7, 8]. The copies from the EQCM are approximate due to fundamental *no-cloning principle* [9]. Each copy is given by the mixed state

$$\rho_k^{\text{out}} = \eta(N, \infty) |\psi_k\rangle \langle \psi_k| + \frac{1}{2} [1 - \eta(N, \infty)] I, \quad (5)$$

where

$$\eta(N, M) = 2^{M-N} \frac{\sum_{l=0}^{N-1} \sqrt{C_l^N C_{l+1}^N}}{\sum_{j=0}^{M-1} \sqrt{C_j^M C_{j+1}^M}} \quad (6)$$

denotes the shrinking factor and  $I$  is the identity operator. Having two infinite ensembles of states  $\rho_a^{\text{out}}$  and  $\rho_b^{\text{out}}$  after the first stage, one can perform measurements in some basis and estimate these states by computing statistical averages. The measurement procedure gives the second stage of the fidelity estimation. Knowing the estimated states we can calculate the fidelity (2). In the line of this strategy, the probability to reconstruct the original state  $|\psi_k\rangle$  from the approximate copies  $\rho_k^{\text{out}}$  is given by  $f_{EQCM} = \langle \psi_k | \rho_k^{\text{out}} | \psi_k \rangle = [1 + \eta(N, \infty)] / 2$ . Therefore, the probability to reconstruct the fidelity (2) correctly equals  $f_{EQCM}^2$ .

In fact, two discussed strategies for the fidelity estimation are equivalent in the sense that the probabilities  $\bar{f}^2(|\psi_a\rangle, |\psi'_a\rangle)$  and  $f_{EQCM}^2$  are equal. This is not surprising and proves the fundamental link between quantum cloning and state estimation [10].

The two strategies above are based on independent estimation of quantum states and computation of the fidelity using the estimated states. However, to estimate the fidelity (2) we do not need to know phases  $\phi_a$  and  $\phi_b$  of the equatorial states (1), rather, the difference between them. Based on this simple observation we now introduce the third two-stage strategy for the fidelity estimation which unifies previous two strategies in some sense. At the first stage one takes a pair of qubits  $|\psi(\phi_a)\rangle$  and  $|\psi(\phi_b)\rangle$  from different ensembles and perform a unitary transformation

$$|\psi(\phi_a)\rangle |\psi(\phi_b)\rangle |\mathbf{A}\rangle_d \longrightarrow |\psi(\phi_a)\rangle |\psi(\phi_b - \phi_a)\rangle |\mathbf{B}\rangle_d \quad (7)$$

on these unknown input qubits. Here  $|\mathbf{A}\rangle_d$  and  $|\mathbf{B}\rangle_d$  are states of an auxiliary system before and after the transformation respectively. The matter of the first stage is to obtain a qubit in the state  $|\psi(\phi_b - \phi_a)\rangle$  at the output of the transformation (7). At the second stage, the state estimation of this state is to be performed what allows us to access information about the phase  $\phi_b - \phi_a$  and by implication to compute the fidelity (2).

Unfortunately, the transformation (7) can not be performed exactly on unknown quantum states. This was first pointed out by Pati and is known today as *the general impossibility theorem* [11]. Recently, however, an optimal approximation to this transformation, the so-called universal CNOT gate, was suggested [12]. This universal gate was obtained by analytical optimization of a general completely positive map applied to two pure equatorial qubit states mediated by an auxiliary system. In construction of the CNOT gate, the standard method for optimization of cloning transformations was employed [8, 13, 14]. The details about the construction of the CNOT transformation and its explicit form can be found in Ref. [12]. For our present discussion, it is important to note that this universal transformation has similar structure to the  $1 \rightarrow 2$  EQCM. The output states of the CNOT transformation are in the mixed states of the form (5) with  $\eta(1, 2)$  and  $|\psi_k\rangle = \{|\psi(\phi_a)\rangle, |\psi(\phi_b - \phi_a)\rangle\}$ . The probability to obtain the idealized outputs from the actual output states of the transformation (7) equals  $f_{CNOT} = 1/2 + 1/\sqrt{8}$ .

Repeating the CNOT gate  $N$  times on available copies of the states  $|\psi(\phi_a)\rangle$  and  $|\psi(\phi_b)\rangle$  we have an ensemble of  $N$  qubits in the mixed state (5) with  $\eta(1, 2)$  and  $|\psi_k\rangle \equiv |\psi(\phi_b - \phi_a)\rangle$  at the output. Having this ensemble we can start the second stage – the state estimation. In general, state estimation of mixed states with unknown shrinking factor and phase require

construction of a specific POVM [15]. However, in our case the shrinking factor is known and, therefore, the state estimation of the mixed state reduces to the estimation of the phase of the pure state  $|\psi(\phi_b - \phi_a)\rangle$ . As we discussed earlier, this task can be accomplished with the POVM (3). Thus the probability to reconstruct fidelity (2) is given by  $\bar{f}(|\psi\rangle, |\psi'\rangle) \times f_{CNOT}$ . This probability is better than the probability to reconstruct fidelity within the measurement-based strategy only for ensembles consisting of single particle. For ensembles of several particles the first strategy becomes more efficient.

The reason for the very limited advantage of the third (unified) strategy over the measurement-based strategy is clear: we applied the universal transformation (7) only on pairs of qubits from different ensembles. A much better strategy is to apply a collective transformation on *all* states in two ensembles at the first stage, i.e.

$$\begin{aligned} |\psi(\phi_a)\rangle^{\otimes N} |\psi(\phi_b)\rangle^{\otimes N} |\mathbf{A}\rangle_d &\longrightarrow \\ |\psi(\phi_a)\rangle^{\otimes N} |\psi(\phi_b - \phi_a)\rangle^{\otimes N} |\mathbf{B}\rangle_d. \end{aligned} \quad (8)$$

Due to the general impossibility theorem, this transformation (8) can not be accomplished exactly on unknown quantum states. However, the optimal approximation for this transformation was recently proposed [16]. The optimal transformation can be obtained using the same technique for optimization of a completely positive map as in case of the transformation (7) and cloning transformations [8]. The optimal approximation for the transformation (8) has similar structure to  $N \longrightarrow 2N$  EQCM. The output states of the transformation (8) are in the mixed states of the form (5) with  $\eta(N, 2N)$ . The probability to reconstruct the idealized outputs from the actual output states is given by  $f_{GCNOT} = [1 + \eta(N, 2N)]/2$ . Coming to the second stage of the fidelity estimation, i.e. performing the state estimation on the ensemble of  $N$  output qubits  $\rho^{out}(\phi_b - \phi_a)$  with POVM (3), we obtain that the probability to reconstruct fidelity (2) equals  $\bar{f}(|\psi\rangle, |\psi'\rangle) \times f_{GCNOT}$ . As displayed in Fig. 2, this probability is always superior to the probability of the fidelity estimation by the measurement-based strategy.

During the discussion of the transformation (8) we paid attention only on one of the output ensembles, i.e. on the ensemble consisting of qubits in the state  $\rho^{out}(\phi_b - \phi_a)$ . One may ask how much information about the state of qubits can be extracted from the other output ensemble of  $N$  particles in the state  $\rho^{out}(\phi_a)$ . The states of the qubits in this ensemble can be estimated with POVM (3). The probability to estimate the state  $\rho^{out}(\phi_a)$  is given

by  $\bar{f}(|\psi\rangle, |\psi'\rangle) \times f_{GCNOT}$ . But, the state  $|\psi(\phi_a)\rangle$  of the initial ensemble of pure equatorial qubits can be estimated much better with probability  $\bar{f}(|\psi\rangle, |\psi'\rangle)$ , if the POVM (3) is applied to this ensemble. Therefore, applying the transformation (8) to the given finite ensembles of equatorial qubits we gain more information about the difference of phases  $\phi_b - \phi_a$  than in case of independent estimation of these phases and, at the same time, obtain less information about the single phase  $\phi_a$  comparing to state estimation of a single ensemble. Other words the information that can be extracted by measurements about a complex quantum system (consisting of the two ensembles of equatorial qubits) is somehow conserved independently on the strategy for the information extraction [17].

At the beginning of the discussion we assumed that both ensembles contain *equal number* of particles in *separable* and *pure* states. In fact, the first assumption can be easily removed. It is easy to define the third strategy for two ensembles with unequal number of particles  $N$  and  $K$ . By analogy with transformation (8), for instance, a generalized  $N \rightarrow N + K$  transformation can be defined [16]. We checked that the third strategy remains the best among the three in the case of unequal number of particles in the ensembles.

However, the other two assumptions are indeed crucial for present discussion. Being given two ensembles of correlated qubits or qubits in mixed states, one should accordingly revise all three strategies. For example, without any knowledge about the shrinking factor of given mixed states, one should use an optimal set of POVM for state estimation of unknown mixed states as it was derived by Bagan *et al.* [15]. Moreover, to apply the third strategy on two ensembles of correlated qubits or qubits in mixed states one should find an optimal approximation for the transformation (8) for such ensembles. It remains an open problem for us whether the third strategy is still the best in the cases of two finite ensembles of unknown equatorial correlated qubits or qubits in mixed states.

In principle, presented analysis of the fidelity estimation problem may be repeated for two finite ensembles of arbitrary qubit states. However, an arbitrary qubit state is characterized by two phases. This leads to a definition of the fidelity between two finite ensembles of qubits that includes four phases, i.e. twice more parameters than the fidelity (2) for equatorial qubits. Within the third strategy, moreover, a transformation that allows us to access the difference between the four phases need to be constructed. Although the desired transformation has not been constructed yet, several indications have been obtained that such a transformation exists.

In conclusion, we have analyzed the three possible strategies for the fidelity estimation between two finite ensembles of unknown pure equatorial qubit states. We showed that the best strategy for the fidelity estimation includes an optimal universal transformation (8) of *all* qubits and the state estimation of the outputs of this transformation by the POVM (3).

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